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Linear interpolation and Sobolev orthogonality☆

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Abstract

There is a strong connection between Sobolev orthogonality and Simultaneous Best Approximation and Interpolation. In particular, we consider very general interpolatory constraints x_i^*, defined by

$$x_i^*(f) = \int_a^b \left(\sum_{j=0}^{n-1} a_{ij}(t) f^{(j)}(t) \right) \mathrm{d}t + \sum_{j=0}^{n-1} \sum_{k=0}^{m} b_{ijk} f^{(j)}(t_k), \quad 0 \le i \le n-1,$$

where f belongs to a certain Sobolev space, $a_{ij}(\cdot)$ are piecewise continuous functions over $[a, b]$, b_{ijk} are real numbers, and the points t_k belong to $[a, b]$ (the nonnegative integer m depends on each concrete interpolation scheme). For each f in this Sobolev space and for each integer l greater than or equal to the number of constraints considered, we compute the unique best approximation of f in $\mathbb{P}_l$, denoted by p_f, which fulfills the interpolatory data $x_i^*(p_f) = x_i^*(f)$, and also the condition that $p_f^{(n)}$ best approximates $f^{(n)}$ in $\mathbb{P}_{l-n}$ (with respect to the norm induced by the continuous part of the original discrete–continuous bilinear form considered).

Keywords: Sobolev orthogonality; Interpolation; Best approximation

1. Introduction

We first state some notations and conventions.

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Throughout this paper, the set of real numbers will be denoted by $\mathbb{R}$, and the set of positive integers will be denoted by $\mathbb{N}$. We will assume that all polynomials considered are real-valued in one real variable, and the set of all such polynomials will be denoted by $\mathbb{P}$. For a nonnegative integer n, $\mathbb{P}_n$ will stand for the subset of $\mathbb{P}$ of all polynomials of degree not greater than n. By a *linear functional* on $\mathbb{P}$ we will mean a function $x_i^*(\cdot) : \mathbb{P} \to \mathbb{R}$ such that $x_i^*(r_1 p_1 + r_2 p_2) = r_1 x_i^*(p_1) + r_2 x_i^*(p_2)$ for all $r_1, r_2 \in \mathbb{R}$ and all $p_1, p_2 \in \mathbb{P}$. The set of all linear functionals on $\mathbb{P}$ (the so-called *dual space of* $\mathbb{P}$), will be denoted by $\mathbb{P}^*$. For $n \in \mathbb{N}$, a (square) matrix of order n, with entries a_{ij}, will be denoted by $A = (a_{ij})_{i,j=0}^{n-1}$, and $(x_i)_{i=0}^{n-1}$ will stand for the matrix of order $1 \times n$ (equivalently, for the vector) $(x_0, x_1, \ldots, x_{n-1})$, whose transpose will be denoted by using the superscript t, that is, by $(x_i)_{i=0}^{n-1\,t}$. As usual, we will identify the only element of a matrix of order 1 with the matrix itself, so the matrix $(x_i)_{i=0}^{n-1}(a_{ij})_{i,j=0}^{n-1}(x_i)_{i=0}^{n-1\,t}$ will be identified with its unique entry $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_{ij} x_i x_j$. The Kronecker delta will be denoted by δ_{ij}, $[\cdot]$ will denote the integer part function (for $x \in \mathbb{R}$, $[x]$ is the greatest integer less than or equal to x), $m \bmod n$ will stand for the remainder on division of m by n, and for $n \in \mathbb{N} \cup \{0\}$, $(\cdot)_n$ will denote the so-called Pochhammer symbol, defined by $(x)_0 = 1$ and $(x)_{n+1} = x(x+1)\cdot\cdots\cdot(x+n)$, for $x \in \mathbb{R}$. We recall that a linear functional u on $\mathbb{P}$ is said to be *regular* (also, *quasi-definite*) if for all positive integers n, the matrix $(\langle u, x^{i+j}\rangle)_{i,j=0}^{n-1}$ is nonsingular (here, the duality bracket $\langle u, \cdot\rangle$ stands for $u(\cdot)$). The regularity of a linear functional u on $\mathbb{P}$ is a necessary and sufficient condition to ensure the existence of a sequence of orthogonal polynomials $\{P_m\}_{m=0}^{\infty}$ with respect to u (fulfilling, then, that $\deg P_m = m$ and $\langle u, P_m P_n\rangle = \kappa_n \delta_{mn}$, where $\kappa_n \neq 0$). MOPS will be the abbreviation for *monic orthogonal polynomial system*.

Now, we show how Sobolev orthogonality has been a fundamental tool to state the orthogonality for some families of classical orthogonal polynomials with non-classical parameters.

In 1995 K.H. Kwon and L.L. Littlejohn (see [11, Theorem 3.1]), stated that for each $k \in \mathbb{N}$, the Laguerre polynomials $\{L_m^{(-k)}\}_{m=0}^{\infty}$ form an orthonormal sequence with respect to a positive-definite inner product defined by the discrete–continuous bilinear form

$$(p, q)_{\mathcal{S}} = (x_i^*(p))_{i=0}^{n-1} A (x_i^*(q))_{i=0}^{n-1\,t} + \left\langle u, p^{(n)} q^{(n)} \right\rangle, \quad p, q \in \mathbb{P}, \tag{1}$$

where $n = k$, $x_i^*(p) = p^{(i)}(0)$ for $i = 0, 1, \ldots, k-1$ (i.e., the linear functionals appearing in the Taylor interpolation scheme), A is a symmetric square real matrix of order k and u is the regular linear functional defined by $\langle u, p\rangle = \int_0^\infty p(x)\mathrm{e}^{-x}\mathrm{d}x$.

A year later, T.E. Pérez and M.A. Piñar gave an elegant and unified approach to the orthogonality of the (generalized) Laguerre polynomials $\{L_m^{(\alpha)}\}_{m=0}^{\infty}$ for every value of the parameter α (see [13]). The technique of this paper, adapted to the discrete case, was used to develop the non-standard orthogonality for Meixner polynomials (see [5]).

In 1998, M. Álvarez, T.E. Pérez and M.A. Piñar stated that for a given positive integer number N, the generalized Gegenbauer polynomials $\{C_m^{(-N+1/2)}\}_{m=0}^{\infty}$ are orthogonal with respect to a Sobolev inner product defined by (1) (see [4]), where now $n = 2N$, where $x_i^*(f) = f^{(i \bmod N)}((-1)^{[i/N]})$, $0 \le i \le 2N-1$ (that is, the linear functionals appearing in two point Taylor interpolation scheme), and where u is the regular linear functional on $\mathbb{P}$ defined by $\langle u, p\rangle = \int_{-1}^{1} p(x)(1-x^2)^N \mathrm{d}x$.

M. Alfaro, T.E. Pérez, M.A. Piñar and M.L. Rezola made, in 1999, an extensive analysis (see [3]) of a Sobolev discrete–continuous bilinear form defined as (1), where now $n = N$ and, for a fixed real number c, $x_i^*(p) = p^{(i)}(c)$, $i = 0, 1, \ldots, N-1$. These authors stated the orthogonality of the families of generalized Jacobi polynomials $\{P_m^{(\alpha,-N)}\}_{m=0}^{\infty}$ and $\{P_m^{(-N,\beta)}\}_{m=0}^{\infty}$, for a fixed $N \in \mathbb{N}$, and real parameters α and β such that $\alpha + N$ and $\beta + N$ are not negative integers. The paper by Alfaro et al. extends and generalizes the study made by I.H. Jung, K.H. Kwon and J.K. Lee in [10].

In the first paper, in 1999, E.M. García-Caballero, T.E. Pérez and M.A. Piñar (see [9]) analyzed the orthogonal polynomials with respect to the bilinear form defined again by means of (1), where now $n = N$ and $x_i^*(p) = p(c_i)$, for $i = 0, 1, \ldots, N-1$ ($c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers). The novelty of this paper consists of the interpolation type and approximation type results that connect Sobolev orthogonality with Best Approximation Theory. In a second paper, in 2000 (see [8]), the same authors generalized previous results by considering the Hermite interpolation scheme in the discrete part of the bilinear form (1), in such a way that with their conclusions, all the orthogonality results mentioned in this historical introduction can be derived (as also the one obtained two years later in [2], involving Jacobi polynomials with negative integer parameters; by the way, the same problem, in a different setting and with a different approach is considered in [12]).

Taking a look at the previous approaches, it seems clear that it is convenient to consider for a fixed positive integer n, a general discrete–continuous bilinear form in which the discrete part contains a vector-valued function $x^* : \mathbb{P} \to \mathbb{R}^n$, defined by $x^*(\cdot) = (x_0^*(\cdot), x_1^*(\cdot), \ldots, x_{n-1}^*(\cdot))$, where for each $i = 0, 1, \ldots, n-1$, x_i^* is a linear functional on $\mathbb{P}$. The linear functionals x_i^* considered are interpolatory constraints of the form

$$x_i^*(p) = \int_a^b \left(\sum_{j=0}^{n-1} a_{ij}(t) p^{(j)}(t) \right) \mathrm{d}t + \sum_{j=0}^{n-1} \sum_{k=0}^{m} b_{ijk} p^{(j)}(t_k), \qquad p \in \mathbb{P},$$

where $a_{ij}(\cdot)$ are piecewise continuous functions over $[a, b]$, $m + 1$ is the number of points in which evaluations of the derivatives of the polynomial p must be done when we evaluate the linear functionals x_i^*, b_{ijk} are real numbers, and the points t_k belong to $[a, b]$.

The structure of the paper is the following: In Section 2 we introduce the symmetric bilinear form on the linear space of real polynomials defined by (1), where A is a square real matrix of order n with all of its principal submatrices nonsingular and u is a regular linear functional on $\mathbb{P}$. Then, we study the relation between the orthogonal polynomials Q_m, associated with this discrete–continuous bilinear form, and the orthogonal polynomials P_m, associated with u. In Section 3 we show that for $m \geq n$, the polynomials Q_m can be expressed as the interpolation error of an nth primitive of $(m - n + 1)_n P_{m-n}$. Then, in the particular case when $\{P_m\}_{m=0}^{\infty}$ is a family of classical orthogonal polynomials, and for any set of n linear functionals $\{x_i^*\}_{i=0}^{n-1} \subset \mathbb{P}^*$, such that their restrictions $x_i^*\big|_{\mathbb{P}_{n-1}}$ are linearly independent, we can construct a Sobolev discrete–continuous bilinear form whose associated orthogonal polynomials Q_m are, for $m \geq n$, the interpolation error of P_m, that is, $Q_m = P_m - \sum_{i=0}^{n-1} x_i^*(P_m)\, l_i$, where $l_0, l_1, \ldots, l_{n-1}$ are the fundamental polynomials for the interpolation scheme whose constraints are the functionals $x_0^*, \ldots, x_{n-1}^*$. In Section 4, after introducing some assumptions on the functionals x_i^*, the matrix A and the linear functional u of (1) (in order to ensure that the bilinear form is, in fact, an inner product), and after considering a wider class of functions (concretely, a Sobolev space denoted by W_2^n), we give a simultaneous best approximation and interpolation type result, since the norm $\|\cdot\|_{\mathcal{S}}$ has the remarkable property that the best approximation of $f \in (W_2^n, \|\cdot\|_{\mathcal{S}})$ in $\mathbb{P}_l$, denoted by p_f, fulfills the n interpolatory constraints $x_i^*(p_f) = x_i^*(f)$, $i = 0, 1, \ldots, n-1$, and also that $p_f^{(n)}$ best approximates $f^{(n)}$ in $\mathbb{P}_{l-n}$ with respect to the norm $\|\cdot\|_{\mathcal{C}} = \sqrt{\langle u, (\cdot)^2 \rangle}$.

2. The Sobolev discrete–continuous bilinear form

Given a fixed positive integer n, let $(\cdot,\cdot)_{\mathcal{S}} : \mathbb{P}\times\mathbb{P}\to\mathbb{R}$ be the symmetric bilinear form on the linear space of real polynomials defined by

$$(p,q)_{\mathcal{S}} = x^*(p)Ax^*(q)^t + \left\langle u, p^{(n)}q^{(n)}\right\rangle, \quad p,q\in\mathbb{P}, \tag{2}$$

where

(i) $x^*\in(\mathbb{P}^*)^n$, that is

$$x^*(p) = (x_i^*(p))_{i=0}^{n-1} = (x_0^*(p), x_1^*(p), \ldots, x_{n-1}^*(p)),$$

for some linear functionals $x_0^*, x_1^*, \ldots, x_{n-1}^*$ on $\mathbb{P}$, such that the set of restrictions $\{x_i^*|_{\mathbb{P}_{n-1}}\}_{i=0}^{n-1}$ is linearly independent, i.e. $\sum_{i=0}^{n-1} c_i\, x_i^*(p) = 0$ for all $p\in\mathbb{P}_{n-1}$ implies $c_i = 0$ for $i = 0,1,\ldots,n-1$.

(ii) $A = (a_{ij})_{i,j=0}^{n-1}$ is a square real matrix of order n with all of its principal submatrices nonsingular,

(iii) $x^*(q)^t$ stands for the transpose of $x^*(q)$,

(iv) and finally, u is a regular linear functional on $\mathbb{P}$.

We are going to construct a basis $\{l_j\}_{j=0}^{\infty}$ in $\mathbb{P}$ in order to compute the Gram matrix associated with the symmetric bilinear form $(\cdot,\cdot)_{\mathcal{S}}$.

Proposition 1. *Let $\{x_i^*\}_{i=0}^{n-1}\subset\mathbb{P}^*$ be such that the set of restrictions $\{x_i^*|_{\mathbb{P}_{n-1}}\}_{i=0}^{n-1}$ is linearly independent. There exists a set $\{l_j\}_{j=0}^{\infty}\subset\mathbb{P}$ verifying*

$$x_i^*(l_j) = \delta_{ij}, \quad i = 0,1,\ldots,n-1,\ j\geq 0.$$

Moreover, for $k\geq n-1$, the set $\{l_j\}_{j=0}^{k}$ is a basis for $\mathbb{P}_k$.

Proof. First we recall (see [7, p. 288]) that there exists a set $\{l_j\}_{j=0}^{n-1}\subset\mathbb{P}_{n-1}$ such that

$$x_i^*(l_j) = \delta_{ij}, \quad i,j = 0,1,\ldots,n-1. \tag{3}$$

To see this, let $\{e_k\}_{k=0}^{n-1}$ be the canonical basis for $\mathbb{P}_{n-1}$ (that is, $e_k(t) = t^k$ for $k = 0,1,\ldots,n-1$). If $\{x_i^*|_{\mathbb{P}_{n-1}}\}_{i=0}^{n-1}$ is linearly independent, then the linear system

$$\sum_{i=0}^{n-1}\alpha_i x_i^*(e_k) = 0, \quad k = 0,1,\ldots,n-1,$$

has a unique solution $\alpha_i = 0$, $i = 0,1,\ldots,n-1$. Therefore the matrix $G = (x_i^*(e_j))_{i,j=0}^{n-1}$ is nonsingular. Then, for each $j = 0,1,\ldots,n-1$, the system of equations

$$\sum_{k=0}^{n-1}\alpha_{kj}x_i^*(e_k) = \delta_{ij}, \quad i = 0,1,\ldots,n-1,$$

has a unique solution $(\alpha_{0j},\alpha_{1j},\ldots,\alpha_{n-1j})$ whose transpose coincides with the $(j+1)$th column of the matrix G^{-1}. Thus, for each $j = 0,1,\ldots,n-1$ the so-called *fundamental polynomials* $l_j = \sum_{k=0}^{n-1}\alpha_{kj}e_k$ belong to $\mathbb{P}_{n-1}$ and verify (3). Note that $(l_k)_{k=0}^{n-1} = (e_k)_{k=0}^{n-1}G^{-1}$. To conclude this first part of the proof, we must show that $\{l_j\}_{j=0}^{n-1}$ is a basis for $\mathbb{P}_{n-1}$, but this is a consequence of the linear independence of the set $\{l_j\}_{j=0}^{n-1}$: If $\sum_{j=0}^{n-1}\alpha_j l_j = 0$ then, for each $i = 0,1,\ldots,n-1$, it follows that

$$\alpha_i = \sum_{j=0}^{n-1} \alpha_j x_i^*(l_j) = x_i^* \left(\sum_{j=0}^{n-1} \alpha_j l_j \right) = 0.$$

Now we define, for each integer $j \geq n$,

$$l_j = \sum_{k=0}^{n-1} -x_k^*(e_j) l_k + e_j \in \mathbb{P}_j \setminus \mathbb{P}_{j-1}.$$

Then, for each $i = 0, 1, \ldots, n-1$ and each integer $j \geq n$,

$$x_i^*(l_j) = \sum_{k=0}^{n-1} -x_k^*(e_j) x_i^*(l_k) + x_i^*(e_j) = 0. \quad \square$$

Now we are going to ensure the existence of a sequence of orthogonal polynomials with respect to the symmetric bilinear form introduced above.

For each positive integer k, let $B_k = (b_{ij})_{i,j=0}^{k-1}$ denote the Gram matrix of order k associated with the regular linear functional u in the basis $\{l_j^{(n)}\}_{j \geq n}$. If we take into account that

(i) for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$,

$$(l_i, l_j)_{\mathcal{S}} = x^*(l_i) A x^*(l_j)^t + \langle u, 0 \rangle = (\delta_{ki})_{k=0}^{n-1} A \, (\delta_{kj})_{k=0}^{n-1\,t} = a_{ij},$$

(ii) for $i = 0, 1, \ldots, n-1$ and each integer $j \geq n$,

$$(l_i, l_j)_{\mathcal{S}} = x^*(l_i) A x^*(l_j)^t + \langle u, 0 \rangle = (\delta_{ki})_{k=0}^{n-1} A (0)_{k=0}^{n-1\,t} = 0,$$

(iii) for each integer $i \geq n$ and $j = 0, 1, \ldots, n-1$,

$$(l_i, l_j)_{\mathcal{S}} = x^*(l_i) A x^*(l_j)^t + \langle u, 0 \rangle = (0)_{k=0}^{n-1} A (\delta_{kj})_{k=0}^{n-1\,t} = 0,$$

(iv) and finally, for each integer $i \geq n$ and each integer $j \geq n$,

$$\begin{aligned}(l_i, l_j)_{\mathcal{S}} &= x^*(l_i) A x^*(l_j)^t + \left\langle u, l_i^{(n)} l_j^{(n)} \right\rangle = (0)_{k=0}^{n-1} A (0)_{k=0}^{n-1\,t} + \left\langle u, l_i^{(n)} l_j^{(n)} \right\rangle \\ &= b_{i-n\, j-n},\end{aligned}$$

then, for each positive integer k, the Gram matrix of order k, G_k, associated with the symmetric bilinear form $(\cdot, \cdot)_{\mathcal{S}}$, in the basis $\{l_j\}_{j=0}^{\infty}$ is given by

$$G_k = \begin{cases} A_k, & \text{if } k \leq n, \\ \begin{pmatrix} A & 0 \\ 0 & B_{k-n} \end{pmatrix}, & \text{if } k > n, \end{cases}$$

where A_k $(1 \leq k \leq n)$ stands for the kth order principal submatrix of the matrix A. Clearly, for all positive integers k, the matrix G_k is nonsingular. Thus there exists a sequence of monic polynomials $\{Q_m\}_{m=0}^{\infty}$, orthogonal with respect to the symmetric bilinear form (2), which will be called in what follows, the *discrete–continuous Sobolev bilinear form*. In the same spirit, the polynomials Q_m will be called *Sobolev orthogonal polynomials*.

Theorem 2. *Let $\{Q_m\}_{m=0}^{\infty}$ be the MOPS associated with the discrete–continuous Sobolev bilinear form (2) and let $\{P_m\}_{m=0}^{\infty}$ be the MOPS with respect to the associated regular linear functional u.*

(i) *The polynomials $\{Q_m\}_{m=0}^{n-1}$ are orthogonal with respect to the discrete bilinear form*

$$(p, q)_{\mathcal{D}} = x^*(p) A x^*(q)^t, \quad p, q \in \mathbb{P}.$$

(ii) *If* $m \geq n$ *then*
  (a) $x_i^*(Q_m) = 0, i = 0, 1, \ldots, n-1$,
  (b) $Q_m^{(n)} = \frac{m!}{(m-n)!} P_{m-n}$.

Proof. (i) For $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$,

$$\kappa_i \delta_{ij} = (Q_i, Q_j)_{\mathcal{S}} = x^*(Q_i) A x^*(Q_j)^t = (Q_i, Q_j)_{\mathcal{D}}.$$

(ii) Fix an integer $m \geq n$. For each $i = 0, 1, \ldots, n-1$ we have that

$$0 = (l_i, Q_m)_{\mathcal{S}} = x^*(l_i) A x^*(Q_m)^t + \langle u, 0 \rangle = (\delta_{ki})_{k=0}^{n-1} A x^*(Q_m)^t.$$

Therefore $A\, x^*(Q_m)^t = 0$. Noting that A is nonsingular, it follows that this system of equations has the trivial solution only, that is $x_i^*(Q_m) = 0$ for $i = 0, 1, \ldots, n-1$, proving part (a).

In case that $i \geq n$ and $j \geq n$, we get

$$\kappa_i \delta_{ij} = (Q_i, Q_j)_{\mathcal{S}} = \left\langle u, Q_i^{(n)} Q_j^{(n)} \right\rangle$$

which means that the polynomials $\{Q_m^{(n)}\}_{m=n}^{\infty}$ are orthogonal with respect to the linear functional u. Thus $Q_m^{(n)} = c_m P_{m-n}$, and a simple inspection of the leading coefficients states that $c_m = m!/(m-n)!$. □

We can give a converse of the previous result, which could be considered as a *Favard-type* theorem.

Theorem 3. *Let* $\{P_m\}_{m=0}^{\infty}$ *be the MOPS associated with a regular linear functional* $u : \mathbb{P} \to \mathbb{R}$, *let* n *be a fixed positive integer, and let* $\{x_i^*\}_{i=0}^{n-1}$ *be a set of* n *linear functionals on* $\mathbb{P}$ *such that the set of restrictions* $\{x_i^*\big|_{\mathbb{P}_{n-1}}\}_{i=0}^{n-1}$ *is linearly independent. If* $\{Q_m\}_{m=0}^{\infty}$ *is a sequence of monic polynomials satisfying*

(i) $\deg Q_m = m, m \geq 0$,
(ii) $x_i^*(Q_m) = 0, i = 0, 1, \ldots, n-1, m \geq n$,
(iii) $Q_m^{(n)} = \frac{m!}{(m-n)!} P_{m-n}, m \geq n$,

then there exists a quasi-definite and symmetric real matrix A *of order* n, *such that* $\{Q_m\}_{m=0}^{\infty}$ *is the MOPS associated with the Sobolev bilinear form defined by*

$$(p, q)_{\mathcal{S}} = x^*(p) A x^*(q)^t + \left\langle u, p^{(n)} q^{(n)} \right\rangle, \quad p, q \in \mathbb{P}, \tag{4}$$

where $x^*(p) = (x_0^*(p), x_1^*(p), \ldots, x_{n-1}^*(p))$.

Proof. Obviously, the polynomial Q_j, with $j \geq n$, is orthogonal to every polynomial Q_k of degree less than or equal to $n-1$ with respect to a Sobolev bilinear form like (4), containing an arbitrary square matrix A of order n in the discrete part (the first term) and the linear functional u in the continuous part (the second term).

We also note that for each integer $j \geq n$ and each integer $k \geq n$,

$$\begin{aligned}(Q_j, Q_k)_{\mathcal{S}} &= x^*(Q_j) A x^*(Q_k)^t + \left\langle u, Q_j^{(n)} Q_k^{(n)} \right\rangle \\ &= \frac{j!}{(j-n)!} \frac{k!}{(k-n)!} \left\langle u, P_{j-n} P_{k-n} \right\rangle = \frac{j! k!}{(j-n)!(k-n)!} \kappa_{j-n} \delta_{jk},\end{aligned}$$

where $\kappa_{j-n} \neq 0$ and, again, A can be considered as an arbitrary square matrix of order n.

Imposing the rest of the orthogonality conditions we can recover the matrix A that appears in the Sobolev bilinear form (4). If we choose arbitrary nonzero real numbers $\kappa_0, \kappa_1, \ldots, \kappa_{n-1}$, we must have in force that for $j = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, n-1$,

$$\kappa_j \delta_{jk} = (Q_j, Q_k)_{\mathcal{S}} = x^*(Q_j) A x^*(Q_k)^t,$$

for some square matrix A of order n. Hence A must verify the matrix relation

$$(x_j^*(Q_i))_{i,j=0}^{n-1} A (x_i^*(Q_j))_{i,j=0}^{n-1} = (\kappa_i \delta_{ij})_{i,j=0}^{n-1}. \tag{5}$$

The linear functionals x_i^* determine the Lagrange polynomials $l_j \in \mathbb{P}_{n-1}$, which verify that $x_i^*(l_j) = \delta_{ij}$ for $i, j = 0, 1, \ldots, n-1$. Taking into account that both $\{Q_j\}_{j=0}^{n-1}$ and $\{l_j\}_{j=0}^{n-1}$ are bases of $\mathbb{P}_{n-1}$, there exists a nonsingular matrix $C = (c_{ij})_{i,j=0}^{n-1}$ such that for each $j = 0, 1, \ldots, n-1$,

$$Q_j = \sum_{k=0}^{n-1} c_{kj} l_k, \quad \text{or, in matrix form,} \quad (Q_i)_{i=0}^{n-1} = (l_i)_{i=0}^{n-1}\, C.$$

We can determine without effort that

$$x_i^*(Q_j) = \sum_{k=0}^{n-1} c_{kj} x_i^*(l_k) = c_{ij},$$

which implies that relation (5) can be expressed in the form $C^t A\, C = D$, where D is an arbitrary nonsingular diagonal matrix. In view of this, if we define

$$A = \left(C^{-1}\right)^t DC^{-1},$$

then A fulfills the conditions of the theorem. Observe that the matrix A is not unique, because its construction depends on the arbitrary nonsingular diagonal matrix $D = ((Q_i, Q_i)_{\mathcal{S}} \delta_{ij})_{i,j=0}^{n-1}$. □

The above result can be used to state most of the non-standard orthogonality results mentioned in Section 1. As an example, we show the Sobolev orthogonality for the Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n=0}^{\infty}$, for $N \in \mathbb{N}$. To fill up the details, we refer the reader to [4], in which this result was originally stated.

Example 4. For every $\lambda \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, the generalized monic Gegenbauer polynomials $C_m^{(\lambda)}$ are defined by

$$C_m^{(\lambda)}(x) = \frac{m!}{2^m} \sum_{k=0}^{[m/2]} \frac{(-1)^k}{(\lambda + m - k)_k k!(m-2k)!} (2x)^{n-2k}, \quad m \geq 0.$$

Taking derivatives in this explicit representation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} C_m^{(\lambda)}(x) = m C_{m-1}^{(\lambda+1)}(x), \quad m \geq 1.$$

Therefore, given a fixed positive integer N, we have

$$\frac{\mathrm{d}^{2N}}{\mathrm{d}x^{2N}} C_m^{(-N+1/2)}(x) = \frac{m!}{(m-2N)!} C_{m-2N}^{(N+1/2)}(x), \quad m \geq 2N.$$

As shown in [4, Proposition 2, (vi)] or as discussed in [14, p. 65, (4)], for each $m \geq 2N$, the points 1 and -1 are zeros of order N of the polynomials $C_m^{(-N+1/2)}$. Hence, for all $m \geq 2N$ and each integer i such that $0 \leq i \leq N-1$,

$$\left.\frac{\mathrm{d}^i}{\mathrm{d}x^i} C_m^{(-N+1/2)}(x)\right|_{x=-1} = \left.\frac{\mathrm{d}^i}{\mathrm{d}x^i} C_m^{(-N+1/2)}(x)\right|_{x=1} = 0.$$

To adjust the notation to the one of Theorem 3, we define:

(i) $n = 2N$,
(ii) $x_i^*(p) = p^{(i \bmod N)}((-1)^{[i/N]})$, for $0 \leq i \leq 2N-1$,
(iii) $Q_m = C_m^{(-N+1/2)}$, for $m \geq 0$.

Then $Q_m^{(n)} = \frac{m!}{(m-2N)!} C_{m-2N}^{(N+1/2)}$ for $m \geq n$, where $\{C_m^{(N+1/2)}\}_{m=0}^{\infty}$ is the family of classical orthogonal polynomials associated with the weight function ρ defined by $\rho(x) = (1-x^2)^N$, and we have also that $x_i^*(Q_m) = 0$ for $0 \leq i \leq n-1$ and $m \geq n$. The conclusion of Theorem 3 is the existence of a Sobolev discrete–continuous bilinear form (we can give it explicitly) such that the corresponding family of orthogonal polynomials is $\{Q_m\}_{m=0}^{\infty} = \{C_m^{(-N+1/2)}\}_{m=0}^{\infty}$.

In concluding this section we give two examples to illustrate the result of Proposition 1. They both deal with well known interpolatory schemes that cannot be described as particular cases of general Hermite interpolation, as Lagrange, Taylor, two point Taylor or simple Hermite (osculatory) interpolation do (see[6]).

Example 5 (*Abel–Gontscharoff Interpolation*). Fix $n \in \mathbb{N}$ and consider n arbitrary (not necessarily distinct) points t_i. Define the linear functionals x_i^* on $\mathbb{P}$ by

$$x_0^*(p) = p(t_0), x_1^*(p) = p'(t_1), x_2^*(p) = p''(t_2), \ldots, x_{n-1}^*(p) = p^{(n-1)}(t_{n-1}),$$

that is, for $i = 0, 1, \ldots, n-1$, $x_i^*(p) = p^{(i)}(t_i)$, for each $p \in \mathbb{P}$. As easy to check, we have

$$\begin{aligned} l_0(t) &= 1, \\ l_j(t) &= \int_{t_0}^{t} \mathrm{d}u_1 \int_{t_1}^{u_1} \mathrm{d}u_2 \int_{t_2}^{u_2} \mathrm{d}u_3 \ldots \int_{t_{j-1}}^{u_{j-1}} \mathrm{d}u_j, \quad 1 \leq j \leq n-1, \end{aligned}$$

and also, for $j \geq n$,

$$l_j(t) = t^j - \sum_{k=0}^{n-1} (j-k+1)_k t_k^{j-k} l_k(t).$$

Example 6 (*Lidstone Interpolation*). Fix $m \in \mathbb{N}$, let $n = 2m+2$ and let t_0, t_1 be two different points. Define the linear functionals x_i^* on $\mathbb{P}$ by

$$\begin{array}{ll} x_0^*(p) = p(t_0), & x_1^*(p) = p(t_1), \\ x_2^*(p) = p''(t_0), & x_3^*(p) = p''(t_1), \\ \vdots & \vdots \\ x_{2m}^*(p) = p^{(2m)}(t_0), & x_{2m+1}^*(p) = p^{(2m)}(t_1). \end{array}$$

Therefore, for $i = 0, 1, \ldots, n-1$, $x_i^*(p) = p^{(2[i/2])}(t_{i \bmod 2})$ for each $p \in \mathbb{P}$. In the "standard" case in which $t_0 = 0$ and $t_1 = 1$, we recall that the first l_j's polynomials can be computed by

$$l_j(t) = \Lambda_{[j/2]}((1 - j \bmod 2)(1-t) + (j \bmod 2)t), \quad j = 0, 1, \ldots, 2m+1,$$

where Λ_k is the unique polynomial (Lidstone polynomial) of degree $2k+1$ defined by the relations

$$\Lambda_0(t) = t,$$
$$\Lambda_k''(t) = \Lambda_{k-1}(t), \quad k \geq 1,$$
$$\Lambda_k(0) = \Lambda_k(1) = 0, \quad k \geq 1.$$

An alternative recursive definition of Lidstone polynomials by means of kernel functions can be found, for example, in [16]. Other explicit representations of Lidstone polynomials are given by (see [1,15])

$$\begin{aligned}\Lambda_k(t) &= \frac{1}{6}\left(\frac{6t^{2k+1}}{(2k+1)!} - \frac{t^{2k-1}}{(2k-1)!}\right) - \sum_{i=0}^{k-2} \frac{2(2^{2i+3}-1)B_{2i+4}}{(2i+4)!}\frac{t^{2k-2i-3}}{(2k-2i-3)!}\\ &= \frac{2^{2k+1}}{(2k+1)!}B_{2k+1}\left(\frac{1+t}{2}\right), \quad k \geq 1,\end{aligned}$$

where B_{2i+4} is the $(2i+4)$th Bernoulli number and $B_k(t)$ is the kth Bernoulli polynomial. Also, and for the same case in which $t_0 = 0$ and $t_1 = 1$ we have, for $j \geq 2m+2$,

$$l_j(t) = t^j - \sum_{k=0}^{2m+1}\left(\frac{\mathrm{d}^{2[k/2]}}{\mathrm{d}t^{2[k/2]}}t^j\bigg|_{t=k \bmod 2}\right) l_k(t) = t^j - \sum_{k=0}^{m}(j-2k+1)_{2k}\Lambda_k(t).$$

3. Sobolev orthogonal polynomials and interpolation

We will show that for $m \geq n$, the polynomials Q_m can be expressed as the interpolation error of an nth primitive of $(m-n+1)_n P_{m-n}$.

Theorem 7. *Let $\{Q_m\}_{m=0}^{\infty}$ be the MOPS associated with the discrete–continuous Sobolev bilinear form (2) and let $\{P_m\}_{m=0}^{\infty}$ be the MOPS with respect to the associated regular linear functional u. For each $m \geq n$ let $R_m \in \mathbb{P}_m$ be an nth primitive of $(m-n+1)_n P_{m-n}$ (that is, $R_m^{(n)} = (m-n+1)_n P_{m-n}$). We have, for $m \geq n$*

$$Q_m = R_m - \sum_{i=0}^{n-1} x_i^*(R_m)\, l_i.$$

Proof. By Theorem 2 we know that for $m \geq n$

$$Q_m^{(n)} = \frac{m!}{(m-n)!}P_{m-n}.$$

Integrating n times we get $Q_m = R_m + S_m$, where S_m stands for an arbitrary polynomial in $\mathbb{P}_{n-1}$. Therefore

$$Q_m = R_m + \sum_{i=0}^{n-1} c_i\, l_i, \tag{6}$$

where $0 = x_j^*(Q_m) = x_j^*(R_m) + \sum_{i=0}^{n-1} c_i\, x_j^*(l_i) = x_j^*(R_m) + c_j$, for each $j = 0, 1, \ldots, n-1$ and each integer $m \geq n$. Replacing c_i by $-x_i^*(R_m)$ in (6) we get the desired conclusion. □

Reciprocally, we have:

Theorem 8. *Let $\{R_m\}_{m=0}^{\infty}$ be a sequence of monic polynomials such that* $\deg R_m = m$ *for each nonnegative integer m. Fix a positive integer n and let $\{x_i^*\}_{i=0}^{n-1}$ be a set of n linear functionals on $\mathbb{P}$ such that the set of restrictions $\{x_i^*|_{\mathbb{P}_{n-1}}\}_{i=0}^{n-1}$ is linearly independent. Set $\{Q_m\}_{m=0}^{\infty}$ the sequence of polynomials given by*

(i) $Q_m = R_m, m = 0, 1, \ldots, n-1,$
(ii) $Q_m = R_m - \sum_{i=0}^{n-1} x_i^*(R_m)\, l_i, m \geq n.$

If $\{R_m^{(n)}\}_{m=n}^{\infty}$ is an OPS with respect to some regular linear functional u, then there exists a symmetric and quasi-definite real matrix A of order n, such that $\{Q_m\}_{m=0}^{\infty}$ is the MOPS associated with the Sobolev bilinear form defined by (2).

Proof. For each nonnegative integer m, Q_m is a monic polynomial of degree m and, in case that $m \geq n$, we have also that $Q_m^{(n)} = R_m^{(n)}$ and $x_i^*(Q_m) = 0$ for $i = 0, 1, \ldots, n-1$. From Theorem 3 we deduce that $\{Q_m\}_{m=0}^{\infty}$ is the MOPS associated with the Sobolev bilinear form (2), where $A = \left(C^{-1}\right)^t \, D\, C^{-1}$, $C = (x_i^*(R_j))_{i,j=0}^{n-1}$, and D is an arbitrary nonsingular diagonal matrix. □

In the light of the previous theorem, we can give the following interesting example.

Example 9. Interpolation error for classical orthogonal polynomials.

Let $\{P_m\}_{m=0}^{\infty}$ be a family of classical orthogonal polynomials, and let $\{x_i^*\}_{i=0}^{n-1}$ be a set of n linear functionals on $\mathbb{P}$, such that their restrictions $x_i^*|_{\mathbb{P}_{n-1}}$ are linearly independent. Due to the classical character of the polynomials P_m, we know that $\{P_{m+n}^{(n)}\}_{m=0}^{\infty}$ is another family of classical orthogonal polynomials. We can define a Sobolev discrete–continuous bilinear form whose associated orthogonal polynomials Q_m measure, for $m \geq n$, the deviation from P_m to its interpolating polynomial, that is,

$$Q_m = P_m - \sum_{i=0}^{n-1} x_i^*(P_m)\, l_i, \quad m \geq n,$$

where $l_0, l_1, \ldots, l_{n-1}$ are the fundamental polynomials for the interpolation scheme whose constraints are the functionals $x_0^*, \ldots, x_{n-1}^*$.

4. Sobolev orthogonal polynomials and simultaneous best approximation and interpolation

With some assumptions on the function x^*, on the matrix A, and on the linear functional u, defining the bilinear form

$$(p, q)_{\mathcal{S}} = x^*(p) A x^*(q)^t + \left\langle u, p^{(n)} q^{(n)} \right\rangle, \quad p, q \in \mathbb{P},$$

introduced in Section 2, this kind of discrete–continuous Sobolev orthogonality can be related to simultaneous polynomial interpolation and approximation. Before stating the connection mentioned, we give a result that will be useful later.

From the proof of Theorem 3 we know that we can recover the matrix A in the discrete part of the Sobolev bilinear form (2), in terms of the linear functionals x_i^*, the first n polynomials Q_i, and in terms of the corresponding n nonzero real numbers $(Q_i, Q_i)_{\mathcal{S}}$. In fact, $A = \left(C^{-1}\right)^t D C^{-1}$, where $C = (x_i^*(Q_j))_{i,j=0}^{n-1}$ and $D = ((Q_i, Q_i)_{\mathcal{S}}\delta_{ij})_{i,j=0}^{n-1}$. Thus

$$A^{-1} = C D^{-1} C^t,$$

and a simple computation yields

$$A^{-1} = \left(\sum_{k=0}^{n-1} \frac{x_i^*(Q_k)x_j^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}}\right)_{i,j=0}^{n-1}.$$

Proposition 10. *Let $\{Q_m\}_{m=0}^{\infty}$ be the MOPS associated with the discrete–continuous Sobolev bilinear form (2). Then, for each $p \in \mathbb{P}$,*

$$\left(p, \sum_{k=0}^{n-1} \frac{x_i^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}} Q_k\right)_{\mathcal{S}} = x_i^*(p), \quad i = 0, 1, \ldots, n-1. \tag{7}$$

Proof. For $i = 0, 1, \ldots, n-1$ we have that

$$\begin{aligned}\left(p, \sum_{k=0}^{n-1} \frac{x_i^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}} Q_k\right)_{\mathcal{S}} &= x^*(p)Ax^*\left(\sum_{k=0}^{n-1} \frac{x_i^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}} Q_k\right)^t \\ &= x^*(p)A\left(\sum_{k=0}^{n-1} \frac{x_i^*(Q_k)\, x_j^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}}\right)_{j=0}^{n-1\,t}.\end{aligned}$$

Since the entries of the vector $(\sum_{k=0}^{n-1} \frac{x_i^*(Q_k)\, x_j^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}})_{j=0}^{n-1\,t}$ are the entries in the $(i+1)$th column of A^{-1}, then

$$\left(p, \sum_{k=0}^{n-1} \frac{x_i^*(Q_k)}{(Q_k, Q_k)_{\mathcal{S}}} Q_k\right)_{\mathcal{S}} = x^*(p)(\delta_{ij})_{j=0}^{n-1\,t} = x_i^*(p), \quad i = 0, 1, \ldots, n-1. \quad \square$$

From now on, we will make the following assumptions on x^*, A and u in (2):

(i) The real matrix A is symmetric and positive definite,
(ii) The regular linear functional u is positive definite,
(iii) There exist real numbers a, b, with $a < b$, a nonnegative integer number m, piecewise continuous functions $a_{ij} : [a, b] \to \mathbb{R}$, real numbers b_{ijk}, and points t_k lying in $[a, b]$, such that for each $i = 0, 1, \ldots, n-1$, the corresponding linear functional x_i^* can be described by

$$x_i^*(p) = \int_a^b \left(\sum_{j=0}^{n-1} a_{ij}(t)p^{(j)}(t)\right) \mathrm{d}t + \sum_{j=0}^{n-1}\sum_{k=0}^{m} b_{ijk}\, p^{(j)}(t_k), \quad p \in \mathbb{P}. \tag{8}$$

By assumptions (i) and (ii) we can ensure that the bilinear form (2) is an inner product. Moreover, since u is positive definite, then there exists a positive-definite Borel measure μ such that for each $p \in \mathbb{P}$,

$$\langle u, p\rangle = \int_{\mathbb{R}} p(t)\mathrm{d}\mu(t),$$

so we can define the "continuous" inner product in $\mathbb{P}$ by

$$(p,q)_{\mathcal{C}} = \langle u, p\,q\rangle = \int_{\mathbb{R}} p(t)q(t)\mathrm{d}\mu(t), \tag{9}$$

with its corresponding associated norm $\|\cdot\|_{\mathcal{C}} = \sqrt{(\cdot,\cdot)_{\mathcal{C}}}$. Thus, the discrete–continuous Sobolev inner product (2) can be written as

$$\begin{aligned}(p,q)_{\mathcal{S}} &= (p,q)_{\mathcal{D}} + (p^{(n)}, q^{(n)})_{\mathcal{C}}\\ &= x^*(p)Ax^*(q)^t + \int_{\mathbb{R}} p^{(n)}(t)q^{(n)}(t)\mathrm{d}\mu(t), \quad p,q\in\mathbb{P}.\end{aligned} \tag{10}$$

Assumptions (i) and (ii), together with assumption (iii), led us to define the functions $x^*(\cdot) = (x_i^*(\cdot))_{i=0}^{n-1}$ and $\langle u,\cdot\rangle = \int_{\mathbb{R}} \cdot \mathrm{d}\mu$ over a wider class of functions. For this purpose, let us consider the Sobolev space, shortly denoted W_2^n, and defined by

$$W_2^n(I,\mathrm{d}\mu) = \{f\in\mathbb{R}^I : f\in C^{n-1}(I), \quad f^{(n)}\in L_2(I,\mathrm{d}\mu)\},$$

where the interval I is the convex hull of the set $[a,b]\cup\mathrm{supp}(\mu)$. In the inner product space $(W_2^n, (\cdot,\cdot)_{\mathcal{S}})$, equipped with the usual norm defined by $\|\cdot\|_{\mathcal{S}} = \sqrt{(\cdot,\cdot)_{\mathcal{S}}}$, the problem of best approximation by elements of the finite dimensional subspace $(\mathbb{P}_l, (\cdot,\cdot)_{\mathcal{S}})$ has a unique solution. The following result can be considered as a simultaneous best approximation and interpolation type result, since the norm $\|\cdot\|_{\mathcal{S}}$ has the remarkable property that the best approximation of $f\in(W_2^n, \|\cdot\|_{\mathcal{S}})$ in $\mathbb{P}_l$, denoted by p_f, fulfills the n interpolatory constraints $x_i^*(p_f) = x_i^*(f)$, $i = 0,1,\ldots,n-1$, and also that $p_f^{(n)}$ best approximates $f^{(n)}$ in $\mathbb{P}_{l-n}$ with respect to the norm $\|\cdot\|_{\mathcal{C}}$.

Theorem 11. *Let $f\in W_2^n$ and let l be an integer with $l\geq n$. If p_f stands for the best approximation of f in $(\mathbb{P}_l, (\cdot,\cdot)_{\mathcal{S}})$ and q_f stands for the best approximation of $f^{(n)}$ in $(\mathbb{P}_{l-n}, (\cdot,\cdot)_{\mathcal{C}})$, then*

(i) $x_i^*(p_f) = x_i^*(f)$, $i = 0,1,\ldots,n-1$,
(ii) $p_f^{(n)} = q_f$.

Proof. To prove (i) we recall that

$$p_f = \sum_{j=0}^{l} \frac{(f,Q_j)_{\mathcal{S}}}{\|Q_j\|_{\mathcal{S}}^2} Q_j,$$

where $(f,Q_i)_{\mathcal{S}}/\|Q_i\|_{\mathcal{S}}^2$ is called the ith-Fourier coefficient of p_f. Using the reproducing property (7) and taking into account that this property can be extended over the class W_2^n, we get, for $i = 0,1,\ldots,n-1$,

$$\begin{aligned}x_i^*(p_f) &= \left(p_f, \sum_{k=0}^{n-1}\frac{x_i^*(Q_k)}{\|Q_k\|_{\mathcal{S}}^2} Q_k\right)_{\mathcal{S}} = \left(\sum_{j=0}^{l}\frac{(f,Q_j)_{\mathcal{S}}}{\|Q_j\|_{\mathcal{S}}^2}Q_j, \sum_{k=0}^{n-1}\frac{x_i^*(Q_k)}{\|Q_k\|_{\mathcal{S}}^2} Q_k\right)_{\mathcal{S}}\\ &= \sum_{j=0}^{l}\sum_{k=0}^{n-1}\frac{(f,Q_j)_{\mathcal{S}}}{\|Q_j\|_{\mathcal{S}}^2}\frac{x_i^*(Q_k)}{\|Q_k\|_{\mathcal{S}}^2}(Q_j,Q_k)_{\mathcal{S}} = \left(f, \sum_{k=0}^{n-1}\frac{x_i^*(Q_k)}{\|Q_k\|_{\mathcal{S}}^2} Q_k\right)_{\mathcal{S}}\\ &= x_i^*(f).\end{aligned}$$

A direct computation in the Fourier expansion of p_f, together with Theorem 2, gives us

$$p_f^{(n)} = \sum_{j=0}^{l} \frac{(f, Q_j)_{\mathcal{S}}}{\|Q_j\|_{\mathcal{S}}^2} Q_j^{(n)} = \sum_{j=n}^{l} \frac{(f, Q_j)_{\mathcal{S}}}{\|Q_j\|_{\mathcal{S}}^2} Q_j^{(n)} = \sum_{j=n}^{l} \frac{(f^{(n)}, Q_j^{(n)})_{\mathcal{C}}}{\|Q_j^{(n)}\|_{\mathcal{C}}^2} Q_j^{(n)}$$
$$= \sum_{j=0}^{l-n} \frac{(f^{(n)}, P_j)_{\mathcal{C}}}{\|P_j\|_{\mathcal{C}}^2} P_j = q_f,$$

which proves (ii). □

Note that for a nonnegative integer m, if p_f denotes the best approximation of $f \in W_2^n$ in $(\mathbb{P}_m, (\cdot,\cdot)_{\mathcal{S}})$, then for $m \le n-1$

$$p_f = \sum_{j=0}^{m} \frac{(f, Q_j)_{\mathcal{D}}}{\|Q_j\|_{\mathcal{D}}^2} Q_j = \sum_{j=0}^{m} \frac{x^*(f) A x^*(Q_j)^t}{x^*(Q_j) A x^*(Q_j)^t} Q_j,$$

and for $m \ge n$

$$p_f = \sum_{j=0}^{n-1} \frac{x^*(f) A x^*(Q_j)^t}{x^*(Q_j) A x^*(Q_j)^t} Q_j + \sum_{j=n}^{m} \frac{(f^{(n)}, Q_j^{(n)})_{\mathcal{C}}}{\|Q_j^{(n)}\|_{\mathcal{C}}^2} Q_j$$
$$= \sum_{j=0}^{n-1} \frac{x^*(f) A x^*(Q_j)^t}{x^*(Q_j) A x^*(Q_j)^t} Q_j + \sum_{j=0}^{m-n} \frac{j!}{(n+j)!} \frac{\int_{\mathbb{R}} f^{(n)}(t) P_j(t) \mathrm{d}\mu(t)}{\int_{\mathbb{R}} P_j^2(t) \mathrm{d}\mu(t)} Q_{n+j}.$$

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